Thesis

Trans-sonic Flow of a

Compressible Fluid

Arthur F. Engelbrecht

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Thesis

# TRANS-SONIC FLOW OF A COMPRESSIBLE FLUID

by

Arthur Frederick Engelbrecht

(A.B., Boston University, 1944)

submitted in partial fulfilment of

the requirements for the degree of

Master of Arts

1948

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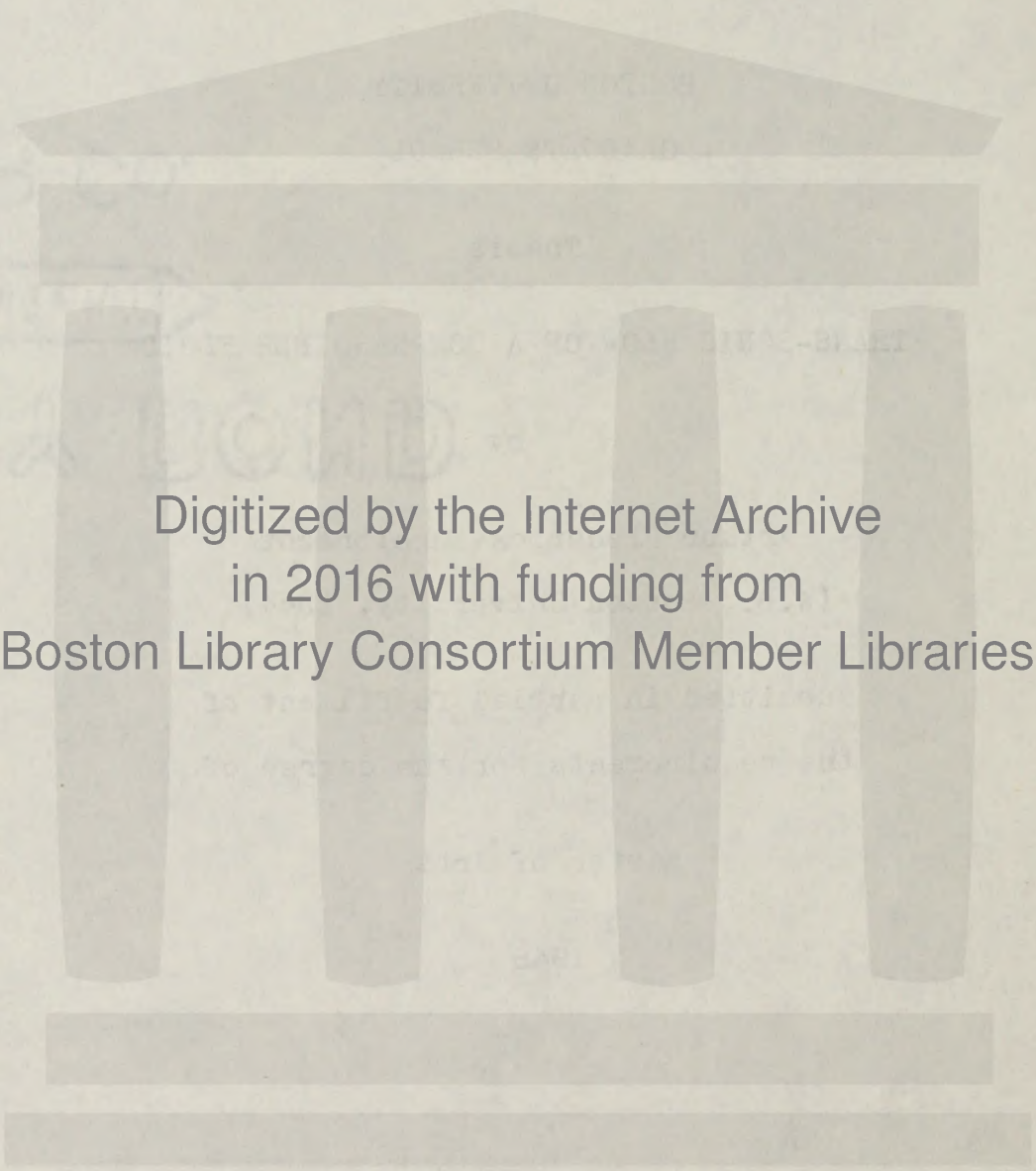

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# TRANS-SONIC FLOW OF A COMPRESSIBLE FLUID

## INTRODUCTION

On stating the problem undertaken, it is well to note that in the theory of fluid flow (hydrodynamic theory) as in general applied mathematical theory, it is customary first to form a mathematical model of the fluid by stating certain axioms or laws to be obeyed and then to draw mathematical deductions of the conduct of the model. The next logical step in the applied theory is to compare deductions from the mathematical model with results of experiments performed on the physical object. If there is a discrepancy between the observations and deductions and the experiment is regarded correct, the deductions are carefully examined and possibly the mathematical model, the postulates and axioms are revised. The purpose of this paper is to draw conclusions from the application of certain axioms and laws to a model fluid, keeping in mind the experimental results.

From photographs of bodies travelling at trans-sonic speed[1], shock waves are actually observed, which waves are not deduced from the mathematically formulated model. The difficulty must be found in the axioms and laws governing the mathematical model.

In the following it is intended to assume the expressions for the conservation of matter, irrotational flow of the fluid,

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[1] von Kármán, T. "The engineer grapples with non-linear problems" Bull. of the Am. Math. Soc. 46 (1940) pp.673,674.

# TRANS-SONIC FLOW OF A COMPRESSIBLE FLUID

## INTRODUCTION

the change of state relation, and the existence of a velocity and stream potential. Using these relations it is possible to obtain differential equations for the velocity or stream potential of the flow. The purpose of the hydrodynamic theory as formulated in this paper is to obtain solutions to these differential equations. It is to be shown that the differential equations obtained for the stream and velocity potentials are of an elliptic character when the velocity of flow is subsonic and of an hyperbolic character when the velocity is supersonic. No actual method for solving such mixed subsonic and supersonic problems has been presented to date and it is not intended to present such a method in this thesis.

The problem is to investigate the various existing solutions for the velocity or stream potential in the case of trans-sonic flow of a compressible fluid, to decide which deductions better suit the model fluid considered, and to indicate the problems which still lie before the compressible flow theory.

As for the importance of the subject matter, flow at high speeds, in the neighborhood of the velocity of sound has been of lay interest for quite some time. That an object arrive at its destination prior to the arrival of the sound wave produced by the object motivated considerable thought before the subject had been thoroughly analyzed.

It is known that projectiles in flight, high speed bullets, attain supersonic velocities, discussion of these having been presented by Lamb in 1923 [2].

G.I.Taylor noted in 1928 [3] that the propeller tips of a plane travelling at four tenths the speed of sound move at 1.3 times the speed of sound.

Also in ballistics, the problem of trans-sonic flow has presented itself in that the terminal velocity of bombs dropped from great heights is in the trans-sonic range.

More recently, chiefly as a result of wartime interest in faster aircraft, rockets, and guided missiles, trans-sonics has become of great importance in physical and mathematical research.

From the point of view of the speed of the fluid, below the speed of sound the solutions to the differential equations of flow are well known and experiment has checked the theory.

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[2] Lamb,Horace et al. Mechanical properties of fluids. p.334, Blackie and Son Ltd. London 1923.

[3] Taylor,G.I. and Sharman,C.F.,"A mechanical method for solving problems of flow of compressible fluids.", Proc.Roy. Soc. A 121:p.194; 1928.

For speeds above the speed of sound, solutions for the velocity and stream potentials may be obtained and it is assumed that in the supersonic range of speeds experiment again checks theory. But, on increasing speed through the speed of sound, very high pressures are experienced with formation of the shock wave on the exposed surfaces of the object in flight. The effect of this shock wave on different surface contours leads to many possible experimental studies. The mathematical formulation of the problem has not yet been completed in a manner which yields satisfactory solutions.

In classical hydrodynamics the fluids are usually assumed incompressible since solutions to the differential equations are more readily obtained with this assumption and these solutions do not differ excessively from the results of experiment when the flow is of subsonic speed. On increasing the speed of flow, the thermodynamic nature of the fluid enters the problem and the compressibility of the fluid must not be ignored.

Study of the flow of a compressible fluid is also not entirely recent, a paper having been presented in the year 1876 by Riemann [4]. The analysis of motion of a compressible fluid through jets or ducts was well presented by Chaplygin in 1902 [5].

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4 Riemann,B.,"Über die Fortpflanzung ebener Luftwellungen von endlicher Schwingungsweite." Gesammelte Werke.,Ges. der Wiss. zu Göttingen. p.144 (1876).

5 Chaplygin,A. "Gas jets" Sci.Memoirs of the Univ. of Moscow, 1902.

It is to be defined at the outset that only macroscopic phenomena are dealt with. In terms of the particle theory of matter, the unit volume element considered shall contain a large number of particles and be assumed continuous.

The fluid under study is formed of continuous matter which may move about freely and thus be readily subject to the action of pressures. More definitely, any continuous matter which offers no resistance to a change in shape with no corresponding change in volume is to be considered a fluid. Thus, liquids and gases qualify for treatment under the definition given above.

That the fluid be compressible, capable of being compressed, refers to the change in volume accompanying a change in pressure on the fluid. In terms of the internal energy of the fluid, isothermal and adiabatic compressibilities are usually distinguished. The isothermal compressibility is the fractional decrease of volume per unit increase of pressure at constant temperature and the adiabatic quantity is the fractional decrease of volume per unit increase of pressure when there is no heat flow. We assert that the sound wave in air travels so rapidly that there is not time enough in the process for a heat transfer, and thus the expression for the adiabatic compressibility of the fluid will be required. For elaboration of the chemical definitions reference is made to to Slater [6].

[6] Slater, J.C. "Introduction to Chemical Physics" p.19, McGraw-Hill Book Co.Inc. New York 1939.

The state of the fluid is defined by assigning to each point in the fluid a velocity vector indicating the velocity of flow at that point. The fluid is said to be in equilibrium when it is in a state of rest. When the fluid is not in equilibrium it is in a state of flow. The state of flow is given by the velocity vectors assigned to the points throughout the fluid. If the flow velocity vector at a given point does not change with respect to time, the state of flow is called the stationary state; if the flow velocity vector does change with time, the flow is in a non-stationary state. The flow is also described by curves called the streamlines of flow, drawn by connecting the flow velocity vectors from point to point, thus forming the tangent curve or streamline of the flow. The streamlines of flow are equal to the lines of motion of the fluid only in the stationary state. The state of flow is also described by a set of curves, related to the streamlines, called the velocity potential lines and defined as the lines of the flow along which the magnitude of the velocity is constant.

The fluid, which is compressible, has elastic properties. It is known that a disturbance in an elastic body will be propagated through the body with a speed dependent on the coefficient of elasticity and the density of the body. In the case of the fluid air, the disturbance is called a sound wave and its magnitude is defined in standard air, that is, air at sea level, ($15^{o}$centigrade, 760 mm. of mercury) as 763 miles per hour [7].

7 Wood, Karl D.,"Technical Aerodynamics" p.1,267. for the figures used.

Pending further definition, the ratio of the speed of an object to the speed of sound in air will be called the Mach number of the flow hereafter. The range of velocities has been arbitrarily divided in terms of the speed of sound in air as follows:

Subsonic range..... up to 600 miles per hour;

Trans-sonic range.... 600 to 900 mph;

Supersonic range..... over 900 mph.

Completing the introduction, the plan of procedure is next presented. In the statement of the aims of applied mathematical theory, it was noted that the deductions on a mathematical model must be confirmed by experiment on the physical object. The first sections of the thesis concern the experimental results, including added descriptions of the flow patterns, means of generating supersonic speeds for observation, and the methods used in measuring the phenomena.

It is interesting to note that the mathematical models prescribed for certain different physical phenomena are alike in form. For example, the mathematical field equations which apply to electrodynamics, mechanics, and hydrodynamics are alike in form. Due to the similarity of the mathematical expressions it is possible to devise analogous experiments from the three physical realms. Whereas it is difficult to perform experiments involving the transonic flow of a compressible fluid, it is much easier to perform experiments analogous to the former by means of an electrolytic bath, or a devised mechanical system. A brief exposition of the experimental analogies used to approximate the trans-sonic flow of a compressible fluid is next given.

The mathematical treatment of the problem follows, starting with the five equations assumed on pages one and two. The non-linear differential equations of the flow are formed and attempts to linearize this equation are presented. The representation of the flow by a stationary, two-dimensional flow in

the physical, hodograph, and logarithmic coordinate systems is presented with criticism. Various series expansions are indicated with the results of the expansions noted.

Sources of data are recorded in the bibliography. A most complete list of printed matter concerning compressible flow theory and non-linear physical theory in general is given by von Karman [8].

8 von Karman, T.,"The engineer grapples with non-linear problems" Bull. of the Am. Math. Soc. 46 (1940)pp.675-683.

## EXPERIMENTAL METHODS

The usual method of examining flow phenomena in aerodynamics is by use of a wind tunnel. The power unit driving the propeller is here of prime importance. A large power unit is required to approach supersonic speeds in a wind tunnel. Usually the pressure in the observation chamber is varied so that supersonic speeds may be more easily attained.

In this way, Dr. Rudolph Hermann achieved speeds of more than four times the speed of sound. His method involved the use of a tunnel which was intermittently connected to a large sphere. The sphere was evacuated and observations on models were made during the time required for the evacuated sphere to be filled with air flowing through the wind tunnel. According to periodicals[9] the German government in the closing days of the second world war was building a wind tunnel of this nature designed to produce speeds near 7,000 miles per hour.

Methods of recording the flow patterns include the surface, pitot, and static pressure tubes familiar to subsonic wind tunnel experiments. The photographic methods used are the Schlieren interferometer, Toepler striation, and the direct shadow methods.[10] The photographic methods rely on the fact

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9 Newsweek Dec. 6, 1946:"German Scientists working for the United States Army" p.64.

10 Fage,A. and Sargent,R.F. "Shock wave and boundary layer phenomena near a flat surface". Proc. Roy. Soc. A. 190-p.1.. June,1947.

that the flow pattern consists of layers of air of varying density and light will be distorted on being transmitted through the fluid. The Toepler striation method of photography yields a very detailed picture of the flow especially in the boundary layers between regions of different density. The direct shadow method yields a well defined image of the shock wave proper with less detail of the flow.

The study of flow patterns for projectiles which travel at trans-sonic speeds indicates the physical problem at hand. At the nose of the projectile, the air undergoes a condensation, a pressure wave is formed. Since air is an elastic fluid, the pressure wave travels from the point of formation in all directions at approximately the speed of sound. For speeds of the projectile below the speed of the pressure wave, the latter travels away from the projectile in all directions. For speeds of the projectile greater than the speed of propagation of sound in air, the pressure wave will not be transmitted in all directions, since only lateral transmissions will be possible. At the nose of the projectile a cushion of air is present. The air in the cushion is compressed causing the projectile to experience much greater pressures when travelling with velocities greater than the velocity of sound.

Referring to flow over the surface of an airfoil, the above named pressure wave is accompanied by the so-called shock wave. This shock wave appears where the direction of the

flow changes. That is, the air in the cushion between the pressure wave and the foil rushes over the surface of the foil's upward curvature at supersonic speeds. On changing directions to the foil's downward curvature, the speed of the flow becomes subsonic as it slows down. The transition from the super- to the subsonic flow is accompanied by a shock wave at the point of transition.

A report on the experimental analogies used to approximate conditions observed at the trans-sonic flow of the compressible fluid now follows.

An electrical analogue is used to solve flow problems for incompressible fluids when the flow is limited to two dimensions. The method consists in passing an alternating current through a layer of liquid of the proper electrolytic conductivity and mapping the amplitude of the alternating voltage. If the height of the layer of electrolyte is variable (the base of the tray may be made of material which may be carved to any desired shape) the density of the fluid analogue (air) may be said to be proportional to the height of the electrolyte. By means of modifying the shape of the bottom of the tray it is possible to solve problems of compressible flow theory by means of an iteration method. The process is applicable only under the initial assumption that the flow of electricity be two dimensional. This assumption is no longer true if the depth of the electrolyte is too great. In problems

of compressible flow the method is not accurate as the velocity of flow approaches the velocity of sound. The use of the electrical analogue thus becomes increasingly erroneous as the velocity approaches the velocity of sound.

A mechanical analogue to two dimensional flow of a compressible fluid of constant depth presents itself in the flow of an incompressible fluid in a channel of gradually varying depth. A consideration of the flow through a duct of variable cross-sectional area may be used to show the essential difference between the mechanics of the flow of a compressible and that of an incompressible fluid.

## MATHEMATICAL FORMULATION OF THE PROBLEM

The limitations imposed on the classical model for hydrodynamic phenomena must be stated and kept in mind in the following pages. The classical theory formulation assumes a continuous space and the presence of force fields derivable from potential functions. From these assumptions solutions to a given flow pattern may be obtained. However, the shock wave, on either side of which different densities of fluid are observed, cannot be completely understood with the classical model. A solution may be approximated if the shock wave be considered a discontinuity in the compressible fluid. Other treatments of the subject consider the flow of a perfect incompressible fluid which is allowed to have a small viscosity and heat conductivity rather than introduce the more advanced thermodynamic relations.

The stepping stones of classical hydrodynamic theory of a compressible fluid are the mathematical expressions for the conservation of matter, the condition of irrotational flow of the fluid, the adiabatic change of state relation, the velocity and the stream potentials of the flow. These five expressions form a set of non-linear differential equations. A transformation of coordinates may be effected whereby the non-linear partial differential equations are linearized and solutions may be obtained to satisfy the mathematical model fluid flow.

It is to be noted that the subject of hydrodynamics, as here considered, exists and is to be developed in plane (and solid) Euclidean space.

Considering now the flow of a compressible fluid, the following notation will be used. Let the orthogonal axes be represented by $x_i$ ; $u_i$ be the components if the velocity of flow U. Two dimensional flow is assumed in the following and therefore the letter subscripts are allowed values 1,2.

The summation notation employed: wherever a subscript appears twice in the same term of the equation, summation is indicated.

As stated before, the following expressions are assumed;

(1) Conservation of matter,

$$(\rho u_i)_{,i} = -\frac{\partial \rho}{\partial t} = \frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2}$$

(2) Irrotational flow of the fluid,

$$(u_i)_{,j} = (u_j)_{,i} \qquad i \neq j$$

(3) Thermodynamic change of state,

$$p = \alpha \rho^{\gamma} + \beta$$

Notation used in (3): $p$ is the pressure of the fluid,

$\rho$ variable density of the fluid,

$\gamma$ adiabatic index, constant for a fluid,

$\alpha, \beta$ constants.

From (1) and vector analysis, it is seen that, in the stationary case, $\rho U$ must be the curl of some function which is

The partial derivative of a quantity with respect to the coordinates is denoted by a comma subscript following the quantity , the subscript indicating the coordinate . See the expression (1) for the Conservation of matter on page 15, as an example.

called the stream function, stream potential, $\Psi$,

(4) $$u_i = \frac{1}{\rho}\Psi_{,j} \quad i \neq j \quad .$$

From vector analysis and (2) it is seen that U must be the gradient of some scalar function which we define the velocity potential, $\emptyset$.

(5) $$u_i = \emptyset_{,i} \quad .$$

In order to set up the equation connecting the density with the velocity of flow, we make use of Bernoulli's relation,

(6) $$\frac{(u_i)^2}{2} + \int_{p_o}^{p} \frac{dp}{\rho(p)} = 0 \quad ,$$

wherein the expression in the denominator of the integral is relation (3). The evaluation of (6) follows, the integral equation is solved by use of relation number 108 in the Peirce integral tables.

From (3) $$\rho = \frac{(p-\beta)^{\frac{1}{\gamma}}}{\alpha^{\frac{1}{\gamma}}} \quad ,$$

$$\int_{p_o}^{p} \frac{dp}{\rho} = \alpha^{\frac{1}{\gamma}} \int_{p_o}^{p} \frac{dp}{(p-\beta)^{\frac{1}{\gamma}}} = \left[\frac{\gamma \alpha^{\frac{1}{\gamma}}}{\gamma-1}(p-\beta)^{\frac{\gamma-1}{\gamma}}\right]_{p_o}^{p}$$

To evaluate the lower limit, we define a stagnation point with values of the quantities, $\rho_o$, $p_o$, $\beta = 0$. Let $\frac{dp_o}{d\rho_o} = c^2$ be the velocity of sound at the defined point.

Since $\alpha = \frac{p_o}{\rho_o^{\gamma}}$ and $p = \frac{p_o}{\rho_o^{\gamma}}\rho^{\gamma} + \beta$ thus, $c^2 = \frac{p_o}{\rho_o}\gamma$.

$$\frac{p_o^{\frac{1}{\gamma}}}{\rho_o}\left[\frac{\gamma}{\gamma-1}(p-\beta)^{\frac{\gamma-1}{\gamma}}\right]_{p_o}^{p} = \frac{p_o^{\frac{1}{\gamma}}}{\rho_o}\left[\frac{\gamma}{\gamma-1}(p-\beta)^{\frac{\gamma-1}{\gamma}} - \frac{\gamma}{\gamma-1}(p_o)^{\frac{\gamma-1}{\gamma}}\right]$$

$$= \frac{c^2}{\gamma-1}\left(\frac{\rho^{\gamma}}{\rho_o^{\gamma}}\right)^{\frac{\gamma-1}{\gamma}} - \frac{c^2}{\gamma-1} \quad ,$$

Substituting into (6) the value of the integral and equating the resultant expression in terms of the density, with the given boundary conditions, (7) is obtained,

$$\frac{u_i u_i}{2} + \frac{c^2}{\gamma - 1}\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} - 1\right] = 0$$

$$\rho = \rho_0\left[1 - \frac{\gamma - 1}{2}\frac{u_i u_i}{c^2}\right]^{\frac{1}{\gamma-1}} \tag{7}$$

The three equations (4), (5), and (7), comprising the equations of motion of the compressible fluid, are seen to form a non-linear differential equation.

In order to attempt a linearization of the non-linear partial differential equations obtained for the flow of a compressible fluid, the following approach to the theory is employed. The effects of volume forces on the system are to be neglected. Volume forces are those forces exerted on the infinitesimal volume elements, such as might be caused by the field of gravity, electricity, or magnetism on the system of fluid. Consideration falls, then, on the area forces, defined as the forces existing between the infinitesimal volume elements and proportional to the area of the face shared by the two volume elements. The area forces in hydrodynamic theory are usually taken to be proportional to the pressure of the fluid.

The equations of motion of the fluid under the above definitions of existing forces are,

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i u_i + p) = 0 \ , \tag{8}$$

Substituting into (6) the value of the integral and equating the resultant expression in terms of the density, with the given boundary conditions, (7) is obtained.

$$\frac{s}{n'n!} + \frac{c_s}{\lambda - 1}\left[\left(\frac{\rho}{\rho_0}\right)^{\lambda-1} - 1\right] = 0$$

$$\rho = \rho_0\left[1 - \frac{s}{\lambda - 1}\frac{n!n!}{c_s}\right]^{\frac{1}{\lambda-1}} \qquad (7)$$

The three equations (4), (5), and (7), comprising the equations of motion of the compressible fluid, are seen to form a non-linear differential equation.

In order to attempt a linearization of the non-linear partial differential equations obtained for the flow of a compressible fluid, the following approach to the theory is employed. The effects of volume forces on the system are to be neglected. Volume forces are those forces exerted on the infinitesimal volume elements, such as might be caused by the field of gravity, electricity, or magnetism on the system of fluid. Consideration falls, then, on the area forces, defined as the forces existing between the infinitesimal volume elements and proportional to the area of the face shared by the two volume elements. The area forces in hydrodynamic theory are usually taken to be proportional to the pressure of the fluid.

The equations of motion of the fluid under the above definitions of existing forces are,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_i}(\rho u_i u_j + p) = 0 \ ,$$

which may be reduced to the following by use of the expression for the conservation of matter, (1),

$$\rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} = 0 \tag{9}$$

The velocity of sound, c, is defined by the relation

$$c^2 = \frac{dp}{d\rho} \tag{10}$$

From the thermodynamic change of state relation (3), and, noting that $\frac{\partial p}{\partial x_i} = \frac{dp}{d\rho} \frac{\partial \rho}{\partial x_i} = c^2 \frac{\partial \rho}{\partial x_i}$, (9) becomes,

$$\rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + c^2 \frac{\partial \rho}{\partial x_i} = 0 \tag{11}$$

Equations (11) and (1) together form a non-linear set of equations of motion of the fluid which will be called (11a),

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{c^2}{\rho} \frac{\partial \rho}{\partial x_i} \tag{11a}$$

$$\frac{\partial (\rho u_i)}{\partial x_i} + \frac{\partial \rho}{\partial t} = 0$$

The first method of linearizing the set of equations (11a) assumes that $u_i$ is small and the deviation of $\rho$ from average value $\rho_0$ is small. These assumptions reduce (11a) to,

$$\frac{\partial u_i}{\partial t} + \frac{c^2}{\rho_0} \frac{\partial \rho}{\partial x_i} = 0 \quad , \quad \frac{1}{\rho_0} \frac{\partial \rho}{\partial t} + \frac{\partial u_i}{\partial x_i} = 0 \tag{12}$$

Eliminating $u_i$ from (12), we obtain the wave equation for $\rho$,

$$\Delta \rho \equiv \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{1}{c^2} \frac{\partial^2 \rho}{\partial t^2} \quad . \tag{13}$$

Under the assumptions made, the equations of motion are linearized, solutions to the wave equation (13) being known. A second method of linearizing equations (11a) makes use of the assumption that $u_2$ be small, $u_1$ differ only slightly from U where U is the mean velocity of flow and may be large. Also, the deviation of $\rho$ from $\rho_0$ is still to be small. Let

$$u_1 = U + u_1' \qquad u_2 = u_2' \quad \text{in (11a), obtaining,}$$

$$\frac{\partial u_1'}{\partial t} + U\frac{\partial u_1'}{\partial x_1} + u_1\frac{\partial u_1'}{\partial x_1} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x_1}, \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1')}{\partial x_1} + U\frac{\partial \rho}{\partial x_1} = 0$$

$$\frac{\partial u_2'}{\partial t} + u_2\frac{\partial u_2'}{\partial x_2} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x_2}, \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_2')}{\partial x_2} = 0$$

From the above and using the assumptions made, obtain (14),

$$\text{(14)} \qquad \frac{\partial u_i'}{\partial t} + U\frac{\partial u_i'}{\partial x_1} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x_i},$$

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x_1} + \rho\frac{\partial u_i'}{\partial x_i} = 0$$

It is now desired to eliminate $u_i$ from both equations (14). This is done by differentiating both equations with respect to $x_i$ and again with respect to t. Differentiate the first equation in (14) with respect to $x_i$, obtaining,

$$\frac{\partial^2 u_i'}{\partial x_i \partial t} + U\frac{\partial^2 u_i'}{\partial x_i \partial x_1} + \frac{c^2}{\rho}\frac{\partial^2 \rho}{\partial x_i^2} = 0;$$

differentiate the second equation in (14) with respect to t,

$$\frac{\partial^2 \rho}{\partial t^2} + U\frac{\partial^2 \rho}{\partial t \partial x_1} + \rho\frac{\partial^2 u_i'}{\partial t \partial x_i} = 0;$$

now differentiating the second equation in (14) with respect to $x_i$ obtain,

$$\frac{\partial^2 \rho}{\partial x_i \partial t} + U \frac{\partial^2 \rho}{\partial x_i \partial x_1} + \rho \frac{\partial^2 u_i'}{\partial x_i^2} = 0$$

From the last two equations on page nineteen,

$$-\frac{\partial^2 u_i}{\partial x_i \partial t} = U \frac{\partial^2 u_i}{\partial x_i \partial x_1} + \frac{c^2}{\rho} \frac{\partial^2 \rho}{\partial x_i^2} = \frac{1}{\rho} \frac{\partial^2 \rho}{\partial t^2} + \frac{U}{\rho} \frac{\partial^2 \rho}{\partial t \partial x_1} .$$

Rearranging the two right-hand members,

$$U \frac{\partial^2 u_i}{\partial x_1^2} = \frac{1}{\rho} \frac{\partial^2 \rho}{\partial t^2} + \frac{U}{\rho} \frac{\partial^2 \rho}{\partial t \partial x_1} - \frac{c^2}{\rho} \frac{\partial^2 \rho}{\partial x_i^2} .$$

Using the expression for $\frac{\partial^2 u_i}{\partial u_1^2}$ from the relation at the top of page twenty, obtain

$$\frac{\partial^2 \rho}{\partial x_i^2} = \frac{1}{c^2} \left[ \frac{\partial^2 \rho}{\partial t^2} + 2U \frac{\partial^2 \rho}{\partial t \partial x_1} + U^2 \frac{\partial^2 \rho}{\partial x_1^2} \right]$$

The quantity in brackets has the form of a quadratic and the expression may be written

$$\Delta \rho = \frac{1}{c^2} \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x_1} \right)^2 \rho \quad . \tag{15}$$

The relation (15) is reducible by an affine transformation to the Laplace equation having solutions for the case $U < c$. Considering the stationary state of the fluid flow (that state of flow for which time derivatives vanish), equation (15) is changed as follows,

$$\left( 1 - \frac{U^2}{c^2} \right) \frac{\partial^2 \rho}{\partial x_1^2} + \frac{\partial^2 \rho}{\partial x_2^2} = 0 \tag{16}$$

The Mach number of the flow was defined previously as the ratio of the speed of flow to the speed of sound. By the definition of U and of c, the Mach number may be defined in terms of these as $M = \frac{U}{c}$. With this value, (16) is written,

(16a) $$(1 - M^2)\frac{\partial^2 \rho}{\partial x_1^2} - \frac{\partial^2 \rho}{\partial x_2^2} = 0$$

This linear differential equation of the second order contains acceptable solutions for values of M below,

(a) $U < c$ $\quad 0 < M < 1$

(b) $U > c$ $\quad M > 1$

In case (a) the differential equation is of the elliptic type and may be reduced to Laplace's equation by the transformation,

$$x_1' = \frac{x_1}{\sqrt{1-M^2}} \quad , \quad x_2' = x_2$$

In case (b), the differential equation is of the hyperbolic type which may be solved by various methods [11].

The two methods of approximation (to linearize equations 11a) herein described do not provide a close analogy to the physical problem in most cases, and other solutions to the non-linear equations must be obtained to better this analogy. The most criticized assumption is that of a mean velocity of flow, U . Imposing a mean velocity on the fluid amounts,

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[11] For a general treatment of both hyperbolic and elliptic differential equations, reference is made to, Courant,R. and Hilbert,D.,"Methoden der Mathematischen Physik" vol.II, 1937 Julius Springer Verlag, Berlin.

usually, to saying that the velocity of flow nowhere differs greatly from the undisturbed velocity, at a distance from the point under observation. This assumption is known to be quite erroneous when the velocity near an object approaches supersonic velocities. In the following section no mean velocity will be specified so that this criticism at least, will be removed.

Using the same notation as before, it is intended to represent the trans-sonic flow by a stationary, irrotational, two dimensional flow. To better distinguish the two sections, the notation for velocities will be changed so that in the following $v_i$ indicates the components of the velocity. Thus, the flow considered must be:

(a) Stationary $\quad \frac{\partial \rho}{\partial t} = \frac{\partial v_i}{\partial t} = 0$ ,

(b) Irrotational $\quad \frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i}$ ,

(c) Two-dimensional $\quad i = 1,2 \quad j = 1,2 \quad i \neq j$ .

Applying condition (a) to the conservation of matter relation (1), we obtain for the continuity equation in our flow,

$$(17) \qquad \frac{\partial v_i}{\partial x_i} = -\frac{v_i}{\rho}\frac{\partial \rho}{\partial x_i} \quad ;$$

evaluate $\frac{\partial \rho}{\partial x_i}$ by use of relation (7) for $\rho$ and (10) for $c^2$ as follows; from (7)

$$\frac{\partial \rho}{\partial x_i} = \rho\left(-\frac{1}{2}\frac{1}{\alpha\gamma\rho^{\gamma-1}}\right)\frac{\partial v_i^2}{\partial x_i}$$

and from (10), $\quad \frac{d\rho}{d\rho} = \alpha\gamma\rho^{\gamma-1} = c^2$ .

Combining the above relations, obtain,

$$(18) \qquad \frac{\partial p}{\partial x_i} = -\frac{\rho}{2c^2}\frac{\partial v_i^2}{\partial x_i} \quad .$$

Equations (17) and (18) are our modified continuity and Bernoulli equations, respectively. Substituting (18) into (17) yields,

$$(19) \qquad \frac{\partial v_i}{\partial x_i} = \frac{v_i v_k}{c^2}\frac{\partial v_i}{\partial x_k} \quad .$$

Using the velocity potential function, $\phi$ , (19) becomes,

$$(20) \qquad \frac{\partial^2 \phi}{\partial x_i^2} = \frac{1}{c^2}\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_k}\frac{\partial^2 \phi}{\partial x_k \partial x_i} \quad ,$$

and, on expanding subscripts i and k, the following is obtained,

$$\frac{1}{c}\frac{\partial \phi}{\partial x_1}\frac{\partial \phi}{\partial x_1}\frac{\partial^2 \phi}{\partial x_1^2} - \frac{\partial \phi}{\partial x_2}\frac{\partial \phi}{\partial x_2}\frac{\partial^2 \phi}{\partial x_2^2} - \frac{\partial \phi}{\partial x_1}\frac{\partial \phi}{\partial x_2}\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\,2 \quad .$$

The cross derivatives are summed by use of the condition that the flow be irrotational. The following is obtained by arranging the above in terms of the second derivatives present,

$$(21) \qquad \left[1 - \left(\frac{v_1}{c}\right)^2\right]\frac{\partial^2 \phi}{\partial x_1^2} - \frac{2v_1 v_2}{c^2}\frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \left[1 - \left(\frac{v_2}{c}\right)^2\right]\frac{\partial^2 \phi}{\partial x_2^2} = 0 \quad .$$

In order to find the character of possible solutions of (21), the primary assumption will be made that (21) is a second ordered differential equation linear in its second derivatives. The assumption on (21) requires that the velocity components $v_i$ be dependent on the coordinates, that is,

$$v_i = f_i(x_1, x_2) \quad , \qquad i = 1,2 \quad .$$

Then we may say, along with the theory of differential equations,

that the type of the differential equation (21) at a point $(x_1,x_2)$ is determined by the value of the determinant of the coefficients of (21). If G is the value of the determinant consisting of elements $g_{ij}$ which are the coefficients of the 'i and j' th partial derivative of $\varnothing$ , then the following conditions determine the character of (21):

if $G > 0$ , the differential equation is of elliptic type,

if $G = 0$ , the differential equation is of parabolic type,

if $G < 0$ , the differential equation is of hyperbolic type.

In the case at hand,

$$G = \begin{vmatrix} 1 - \left(\frac{v_1}{c}\right)^2 & -\frac{v_1 v_2}{c^2} \\ -\frac{v_2 v_1}{c^2} & 1 - \left(\frac{v_2}{c}\right)^2 \end{vmatrix} = 1 - \frac{v_1^2 + v_2^2}{c^2} = 1 - \frac{v^2}{c^2}$$

According to the statements on the type of the differential equation given above, (21) is seen to be of :

(22) elliptic type if $v < c$ ,
hyperbolic type if $v > c$ ,
parabolic type if $v = c$ .

From (22) the difficulties involved in obtaining solutions for the velocity potential in the physical plane are realized. Three different types of equations must be solved as the velocity of flow in the channel passes through the velocity of sound. It is to be remembered that these difficulties have occurred when the flow is described in the physical plane, that is, the frame of reference with coordinates $x_1,x_2$ .

More will be said about other frames of reference. Again, it is proper to investigate the initial assumptions leading to the conclusions (22). It was assumed that the coefficients of the second derivatives in (21) were linear. The degree of error of this assumption should be pointed out.

The value c was defined by (10) to be, $c^2 = \frac{dp}{d\rho}$. By this definition, the speed of sound in an incompressible fluid would be infinite. This infinite value of c reduces (20) to the Laplace equation before mentioned as having been thoroughly treated in potential theory. For compressible fluids, $c^2$ has a finite value and (20) has coefficients dependent of the velocity of flow and the velocity of sound in the fluid. Solutions to (21) in the compressible case, with c finite, are obtained by methods used to solve non-linear differential equations.

Dealing exclusively with similarities in flow patterns, it is possible to set up the pattern for the flow of a compressible fluid past a given object by using the flow pattern obtained for an incompressible fluid past the same body. The empirical methods, using alternating current in an electrolytic solution, or incompressible fluid (water to a close approximation) in a channel of gradually increasing depth, have been described (on pages 12 and 13). As results of experiment, G.I. Taylor and C.F. Sharman have stated [3] that the approximation to the compressible flow potential in the electrical case is not very

good. von Karman states [12] that the variable depth channel method yields qualitative information of the possible flow patterns in the case of supersonic flow of a gas corresponding to "supercritical flow" in the channel.

A study of flow patterns indicate that if the element of an incompressible flow field be taken to be a square shaped element (the distances between two stream potential lines and two velocity potential lines be approximately equal), then the corresponding field assuming a compressible fluid will be made up of rectangular elements. Noting this, Bryan [3] attempted to generalize that, due to the adiabatic compressibility of a compressible fluid, the square shaped elements in the flow pattern of an incompressible fluid are distorted from squares to rectangles. This planned distortion method failed.

A second method used in solving the non-linear partial differential equations consists in representing the potential functions of the trans-sonic flow with the assumption of axial symmetry on the part of the body around which the potential functions are to be determined. The velocity potential about a cylinder and a sphere have been approximated by Lord Rayleigh with certain symmetry assumptions [13], and treatments of other surfaces usually start with these approximations. The assumption that the velocity vector nowhere differ much from the

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[12] von Kármán,T. "Compressibility effects in Aerodynamics". Journal of Aeronautical Science. 8 p 337, July 1937

[13] Lord Rayleigh,"On the flow of a compressible fluid past an object". Phil.Mag. and Journal of Science. 32 p.1. July 1916.

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undisturbed velocity, or mean velocity (as in the development starting on page 19) is equal to the assumption that the body around which the potential functions are being determined be axially symmetric. If the body under investigation presents a small area to the flow, such as a thin airfoil, then the method of small perturbations on the velocity potential is used.

Otherwise, in the symmetrical body treatment, the differential equations (21) may be solved by a series expansion of the velocity potential in terms of powers of the Mach number of the flow. As in the previous work, U shall be the velocity of the undisturbed flow, and the value $M = \frac{U}{c}$ with c the sonic speed in the undisturbed region, is the Mach number of undisturbed flow. The values $\phi$ used in the successive approximations are all assumed functions of the coordinates $x_1, x_2$. Given the series expansion of $\phi$,

$$\phi = U\left[\phi_0 + \phi_1 M^2 + \phi_2 M^4 + \dots\right] \quad . \tag{23}$$

On substitution into (20) note that (23) provides as a first approximation,

$$\Delta \phi_0 = 0 \quad , \tag{24}$$

an expression with the form of the Laplace equation which is the differential equation obtained for the potential of the incompressible fluid. Obtaining the next approximation in terms of $\phi_0$,

$$\Delta \phi_1 = \sum_i \sum_k \frac{\partial \phi_0}{\partial x_i} \frac{\partial \phi_0}{\partial x_k} \frac{\partial^2 \phi_0}{\partial x_i \partial x_k} \quad . \tag{25}$$

The partial derivatives in (25) may be computed from the solutions to the analogous problem for the incompressible fluid, (24), and then substituted in (25). Thus, the first approximation to the velocity potential is expressed as a function of the coordinates $x_1, x_2$. Similarly, having the function $\phi_1$ as dependent on the coordinates, it is possible to obtain the second approximation, $\phi_2$, and express it as a function of position, and so on, to any desired degree of approximation. Thus, the solution to (25) for the given boundary conditions at the surface of the body is made equivalent to obtaining expressions of the velocity potential around a body in an incompressible fluid, in which the potential is produced by a field of continuously distributed sources. The sources of intensity are assumed in the problem when the statement is made that the first approximation to the velocity potential shall be a function of coordinates, i.e., $\Delta \phi_1 = F(x_1, x_2)$.

The method outlined provides reasonable solutions if the flow is around bodies of symmetrical form, cylinders and spheres for example, because the solutions of (25) may be obtained by superposition for these symmetrical bodies. The method is laborious for approximations higher than the first and the series expansion converges very slowly if the Mach number of the flow is greater than 0.4 . Clearly, the series expansion here indicated does not find place in problems of trans-sonic flow where the Mach number is greater than 0.6 .

The method of small perturbations, as indicated above, rises from the assumption that the velocity of flow nowhere differ greatly from the undisturbed velocity. On page 19 and following it was assumed that the density nowhere differ greatly from a mean density. Expressions analogous to those derived on page 21 are obtained for the velocity potential. In the development below it is assumed that the velocity potential must contain an added perturbing potential function which does not differ much from the given velocity potential. Thus, it is assumed that $\phi = Ux + \phi'$, and, analogous to (16),

$$(26) \qquad \left(1 - \frac{U^2}{c^2}\right) \frac{\partial^2 \phi'}{\partial x_1^2} + \frac{\partial^2 \phi'}{\partial x_2^2} = 0$$

In an analysis similar to that proposed on page 21, it is seen possible to reduce (26) to the Laplace equation when $U < c$, in the case for subsonic flow by the transformation,

$$(27) \qquad x_1' = x_1 \qquad x_2' = x_2\sqrt{1 - M^2}$$

If $U > c$, that is, the undisturbed flow is supersonic, (26) again has the form of a wave equation and solutions are known.

Due to the transformation (27) it has been stated that the flow of a compressible fluid around a thin body with Mach number of undisturbed flow $M$, may be found approximately by expanding the dimensions of the body normal to the main flow such that the dimensions in the incompressible flow are multiplied by $\sqrt{1 - M^2}$ to obtain the above stated dimensions in the compressible case. It is to be noted that this method is

still one of obtaining solutions to a differential equation which is non-linear. Also that no discussion is made of the physical case where the Mach number of the flow is 1.0 .

The next representation of flow patterns to be discussed is the transformation from the physical to the hodograph plane, first effected by Molenbroek[14] and Chaplygin[15] . The velocity and stream potentials are here considered functions of the magnitude, v, and the direction, $\theta$ , of the flow velocity vector. It is possible to obtain two linear partial differential equations where the potential functions satisfy the systems,

$$\frac{\partial \varphi}{\partial \theta} = \frac{v}{\rho} \frac{\partial \psi}{\partial v}$$

(28)

$$\frac{\partial \varphi}{\partial v} = \frac{1 - \left(\frac{v}{c}\right)^2}{\rho v} \frac{\partial \psi}{\partial \theta}$$

The new coordinates in terms of the former are, $v = \sqrt{v_1^2 + v_2^2}$ and $\theta = \tan^{-1} \frac{v_2}{v_1}$ .

The density and speed of sound are expressible as functions of the flow velocity and the thermodynamic constants so that it is possible to eliminate them using relations (3), (7), and (10) and the definition of the stagnation point. Lighthill[16] using

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[14] Molenbroek,P. "Über einege Bewegungen eines Gases mit Annahme". Arch.Math.Phys., 9, pp.157-195, 1890.

[15] Chaplygin,S.A. 1904,Ann.Sci.Moscow Imp.Univ.Math.Phys.Sect. 21,1.

[16] Lighthill,M.J."The hodograph transformation in trans-sonic flow" Proc.Roy.Soc.A,191, pp 304-370; November 1947.

the above method, obtains,

$$\frac{\partial \phi}{\partial \theta} = \frac{\partial \psi}{\partial v} \frac{v}{(1 - v^2)^{\frac{1}{\gamma-1}}}$$

(29)

$$\frac{\partial \phi}{\partial v} = \frac{\partial \psi}{\partial \theta} \frac{(\gamma + 1)v^2 - (\gamma - 1)}{(\gamma - 1)v(1 - v^2)^{\frac{\gamma}{\gamma-1}}}$$

$\phi$ is eliminated from the above equations by differentiating the first with respect to $v$ and the second with respect to $\theta$, then equating the two expressions to obtain,

$$(30) \quad \frac{(\gamma-1)(1-v^2)\,v^2}{(\gamma+1)v^2 - (\gamma-1)} \frac{\partial^2 \psi}{\partial v^2} + \frac{2v(\gamma-1)\left[(1-v^2) + \frac{v^2}{\gamma-1}\right]}{(\gamma+1)v^2 - (\gamma-1)} \frac{\partial \psi}{\partial v} - \frac{\partial^2 \psi}{\partial \theta^2} = 0$$

Rearranging the terms in the above,

$$(31) \quad \frac{\partial^2 \psi}{\partial \theta^2} = \frac{(\gamma-1)}{(\gamma+1)v^2 - (\gamma-1)} \left[ v^2(1-v^2) \frac{\partial^2 \psi}{\partial v^2} + 2v\left(1 + \frac{2-\gamma}{\gamma-1} v^2\right) \frac{\partial \psi}{\partial v} \right]$$

is obtained. It is noted that (31) is linear, since the coefficients of the right hand side are functions of the velocity ($\gamma$ is assumed to be the adiabatic constant).

The general solution of (31), regular when $v = 0$ and odd in $\theta$ is

$$(32) \quad \psi(v^2, \theta) = A_0 \theta + \sum_{n=1}^{\infty} A_n \psi_n(v^2) \sin n\theta \quad ,$$

where the $\psi$ under the summation sign indicates the hypergeometric functions,

the above method, obtains,

$$\frac{\partial \mathcal{G}}{\partial \theta} = \frac{\partial \Psi}{\partial v} \frac{v}{(1 - v^2)^{\frac{1}{\gamma-1}}}$$

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where the $\Psi$ under the summation sign indicates the hypergeometric functions,

$$(33) \quad \psi_n(v^2) = v^n F(a_n, b_n; n+1; v^2)$$

where $$a_n + b_n = n - \frac{1}{\gamma - 1}$$

$$a_n b_n = -\frac{n(n+1)}{2(\gamma - 1)}$$

$$a_n, b_n = \frac{1}{2}\left[ n - \frac{1}{\gamma - 1} \pm \sqrt{\frac{(\gamma+1)n^2}{(\gamma-1)} + \frac{1}{(\gamma-1)^2}} \right]$$

The expansion of (33) follows from the definition of the hypergeometric series,

$$(34) \quad \psi_n(v^2) = v^n \left[ 1 + \frac{a_n \cdot b_n}{1 \cdot (n+1)} v^2 + \frac{a_n(a_n+1)\, b_n(b_n+1)}{1 \cdot 2 \cdot (n+1)(n+2)} v^4 + \cdots \right]$$

The analysis of the hypergeometric function is given in Whittaker and Watson [17] including convergence of the series and the residues about the ennumerated poles, important to the theory of flow in the hodograph plane. Lighthill evaluates the subsonic and supersonic expressions for (32) and gives a connected account of the properties of the expansion as a function of n in the complex plane.

At the sonic speed, $v = \sqrt{\frac{\gamma - 1}{\gamma + 1}}$ and at the velocity $v = \infty$ the hypergeometric series expansion fails so that at these points singularities are observed in the hodograph plane.

The following conditions are imposed on the $\psi_n$ in order to obtain a steadily increasing velocity on the axis of the

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[17] Whittaker, E.T. and Watson, G.N. A Course of Modern Analysis. pp.281-301 Nov.1946, MacMillan Company New York.

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[17] Whittaker, E.T. and Watson, G.N. A Course of Modern Analysis. pp. 281-301 Nov. 1946, MacMillan Company New York.

channel from the value 0 at $x = -\infty$ to the value $v_1$, a supersonic value, at $x = +\infty$:

$$\psi_\theta < 0 \quad \text{for} \quad 0 < v < \sqrt{\frac{\gamma - 1}{\gamma + 1}} \qquad \psi_\theta = \infty \text{ at } v = \sqrt{\frac{\gamma - 1}{\gamma + 1}}$$

(35)

$$\psi_\theta > 0 \quad \text{for} \quad \sqrt{\frac{\gamma - 1}{\gamma + 1}} < v < v_1 \qquad \psi_\theta = \infty \text{ at } v = v_1$$

With these conditions imposed on the flow, a linear combination of values (32) represents the flow in a channel with v rising steadily on the axis from 0 to $v_1$. The linear combination may be written in terms of $\theta$, since the sine terms may be expanded in series, thus,

$$\psi = \theta Q_0 + \theta^3 Q_1 + \theta^5 Q_2 + \theta^7 Q_3 + \ldots \tag{36}$$

If (36) diverges, an expression connecting $\psi$, v, and $\theta$ is obtained by inverting (36) obtaining the following series,

$$\theta = \psi R_0 + \psi^3 R_1 + \psi^5 R_2 + \psi^7 R_3 + \ldots \tag{37}$$

The chief advantage of this representation is that (37) converges uniformly in v in any range $v \geq v_0 > 0$ and in particular near the speed of sound, $v = \sqrt{\frac{\gamma - 1}{\gamma + 1}}$.

The above discussion has been for flow in a channel, with no body present. Using the hypergeometric expansion, a very general solution for subsonic flow around a body which reduces to the incompressible flow as the Mach number approaches zero, is possible. The characteristic fault of the analysis in the

subsonic region of flow is that the solution in one part of the flow does not connect with the solution for another region. The solutions in the hodograph representation are valid everywhere, although there is a singularity at the sonic speed.

The representation of formula (33) is quite complicated when the velocity of flow is greater than the speed of sound, that is, when $v>\sqrt{\frac{\gamma-1}{\gamma+1}}$. When the supersonic flow is being calculated around a body, series expansions, of the type of the generalized Laurent expansion, must be used around such symmetrical bodies as circles, ellipses, polygons, or the Joukowski airfoils. The expansions are very difficult to obtain in the cases of more general airfoils.

It is expected that inherent difficulties with the supersonic representation in the hodograph plane will result in attempts to express the flow by other means. Mathematical work on the essentially mathematical problem of obtaining continuous solutions in the different regions of investigation noted above must yet be perfected. As a step in this direction, Bergman[18] notes that it still seems preferable to use the hodograph method, linking with it the method of operators which transform solutions of one partial differential equation into solutions of another one. For, example, it is possible to find operators which will transform the solutions of an equation of elliptic type into solutions of an hyperbolic type.

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[18] Bergman,S. "Two-dimensional subsonic flow of a compressible fluid and their singularities". Trans.Am.Math.Soc.62,452-Nov!47

However, since the solutions of both equations have quite different character, it is stated that the operators cannot preserve various properties of the functions upon which they act. The same article by Bergman listed above, contains a description of the flow in a logarithmic and a pseudo-logarithmic plane. This representation allows the introduction of imaginary and real components of the variables by a transformation of coordinates from $v$ and $\theta$ to $\bar{Z}$ and $\bar{\lambda}$ ,

$$\bar{Z} = i \log \bar{v} \qquad \bar{v} = v_1 + iv_2 = ve^{-i\theta} \tag{38}$$

$$\bar{\lambda} = \log v$$

and $v_1$ , $v_2$ , $\theta$ , are the components and direction respectively of the flow velocity vector.

However desirable the separation of flow into real and imaginary parts may be, it is concluded by Bergman that further work should be done in the hodograph method for better representation of the flow in the different regions.

## CONCLUSION

Flow patterns in various representations have been investigated in the case of the trans-sonic flow of a compressible fluid. The fluid, in addition to being compressible, is assumed to obey the law of conservation of matter, be irrotational in flow and obey the adiabatic change of state relation. The discussion further limits the kind of fluid to air, since we speak throughout of trans-sonic speed, implying the region of the speed of sound in air.

The non-linear partial differential equations of flow of the fluid expressed in the Cartesian (physical) plane have been developed starting on page 17.

On page 18 two attempts at linearization of the non-linear partial differential equations are presented. The assumptions made in the attempts, (a) that the velocity be small, (b) that the velocity nowhere differ much from an undisturbed velocity, both cripple the attempted solutions of the problem in the case of trans-sonic flow.

It is found (p.24) that the velocity potential in the physical plane may, from the differential equations be a solution of an elliptic, parabolic, or hyperbolic differential equation dependent on the velocity of flow being less than, equal to, or greater than the velocity of sound in air.

Flow pattern analysis points out the insufficiency of the

# CONCLUSION

physical plane representation. The attempt to form a flow pattern in the case of a compressible fluid by mathematically altering the flow pattern of a corresponding incompressible fluid was declared unsuccessful (p.26).

On the assumption of axial symmetry, the velocity potential is expanded in terms of the Mach number in order to approach a continuous value for the velocity potential in the trans-sonic region. It is seen that the convergence of the series slows down considerably before the trans-sonic region is neared (p.26).

Perturbation theory is also shown to be inadequate on considering trans-sonic flow of the fluid. The differential equations obtained are still non-linear and the basic assumption that there be no great velocity difference from an undisturbed velocity is not a correct assumption for trans-sonic flow(p.29).

The hodograph transformation, however, as indicated on page 30, yields linear partial differential equations whose solutions are the desired potentials of flow. The stream potential is here evaluated, since it is more easily formulated. Solutions are given in the form of the hypergeometric series. The speed of sound and the speed at infinity are still singularities of the flow. A method is indicated (p.33) by which the velocity potential may be approximated by a series expansion even closeto the speed of sound.

A brief indication of the coordinates used in a logarithmic representation closes the thesis.

Hovering over the problem still is the fact pointed out on page two that no mathematical solutions to non-linear partial differential equations of the type here discussed have yet been presented.

# ABSTRACT OF THE THESIS

At the outset, it is stated that a model fluid is defined as obeying certain laws or axioms from which deductions are to be drawn. The deductions should agree with experiments performed on the fluid.

A model fluid has been created which: (a) is continuous; (b) obeys the law of conservation of matter; (c) is irrotational in flow; (d) obeys the adiabatic change of state relation; (e) may be described by a system of curves, called the velocity and stream potentials of flow. The above statements, not supposed independent of one another, are taken to be the most prominent laws or axioms imposed on the model fluid. The problem at hand is to investigate the behaviour of the model fluid in trans-sonic flow as far as law (e) is concerned. The question to be proposed is how is the model fluid, traveling at trans-sonic speeds, described in terms of the velocity and stream potentials of the flow?

The general term fluid is limited throughout the thesis to air, but the same mathematical discussion follows for any fluid satisfying laws (a) to (e). Also, in the analysis performed, it is assumed that if solutions for both the velocity and stream potential could be obtained, it would be sufficient for the problem to attempt solution for that potential which could be more easily formulated. Thus, in the physical plane, the velocity

potential and in the hodograph plane the stream potentials are investigated.

In the section entitled Experimental Methods, the photographic means of recording flow patterns are given. It is there noted that on passing through the speed of sound, a shock wave is formed, on either side of which different densities of the fluid are observed. This shock wave might be taken as an indication that the fluid at the speed of sound, is undergoing an un-adiabatic change of state and thereby invalidate the law (d) above. Whatever the condition of flow at this speed might be, the mathematical formulation yields singularities which are the subject of the investigation.

The physical plane representation is especially singular at the speed of sound. A set of non-linear partial differential equations is obtained which would yield for the velocity potential solutions of elliptic, parabolic, and hyperbolic character depending on the speed of flow being just below, equal to, or just above the speed of sound.

Attempts to obtain a solution for the velocity potential, which is continuous in the trans-sonic region are developed and criticised. The non-linear differential equation obtained for the velocity potential is linearized by different assumptions which limit the validity of the velocity potential in the trans-sonic region.

The perturbation theory application allows speculation

on the nature of the supersonic flow pattern, but the treatment fails in the trans-sonic region. The series expansion method of approximation of the velocity potential fails here also.

The hodograph representation of the flow yields linear differential equations which are solved in terms of the hypergeometric series. The study of the convergence, singularities and residues of the series has been made (p.32). Approximations for the stream potential are obtained. The solutions remain valid close to the points of singularity of the flow, for, if the approximating series does not converge, the inverse series is taken as the solution.

Thus, in the hodograph plane it is possible to approximate the stream and velocity potentials by using a series expansion based on the hypergeometric series, and these approximations are valid close to the speed of sound.

It is to be noted that in the hodograph plane the singularities at the speed of sound and at infinite speed are still present. The presence of this singularity in the mathematical treatment of the flow indicates that other mathematical methods for solving such mixed problems ( sub- to trans- to supersonic) are to be sought.

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